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AN APPROACH TO SOLVING THREE-DIMENSIONAL DYNAMIC PROBLEMS OF THE THEORY OF ELASTICITY AND VISCO-ELASTICITY FOR BODIES OF COMPLEX SHAPE*

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An approach is proposed for finding the natural frequencies and natural spatial shapes of the oscillations of an elastic body, which is suitable for bodies of a complex geometrical shape. The approach is based on the method of backward iterations /1/, using the method of geometric imbedding at each iteration**, (**See I.N. Shardakov, I.E. Tryanovskii, and I.N. Trufanov, The method of geometric imbedding for solving boundary value problems of the theory of elasticity, Preprint, In-t Mekhaniki Sploshnykh Sred, Sverdlovsk, 1984.). Examples are given of the numerical realization of the iterative algorithm, and the natural frequencies and shapes are found for some bodies of complex geometrical shape. These natural shapes are used as basis functions for studying the forced steady-state oscillations of visco-elastic bodies of complex shape.

1. The determination of the natural shapes and frequencies of the oscillations of a linear elastic body occupying a domain Ω in three-dimensional Euclidean space with boundary Γ, under homogeneous boundary conditions in the displacements on the part Γ_u of the boundary and in the stresses on the remainder Γ_σ of the boundary, reduces to finding the non-trivial solutions of the variational equation

$$\forall \mathbf{v} \in V, \quad \langle \mathbf{u}, \mathbf{v} \rangle = \lambda (\mathbf{u}, \mathbf{v}) \tag{1.1}$$
$$V = \{\mathbf{v} \in (H^1(\Omega))^n; \ \ \mathbf{v} = 0, \ \mathbf{x} \in \Gamma_u\}, \quad n = 3$$
$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \sigma(\mathbf{u}) \cdot\cdot\, e(\mathbf{v})\, d\Omega, \quad (\mathbf{u}, \mathbf{v}) = \int_\Omega \rho \mathbf{u}\mathbf{v}\, d\Omega$$
$$\| \mathbf{u} \| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2}, \quad | \mathbf{u} | = (\mathbf{u}, \mathbf{u})^{1/2}$$

Here, V is the Hilbert space of vector functions, while the last equations define the scalar products and the norms generated by them; σ, e are the stress and deformation tensors, ρ is the material density, λ is the wanted eigenvalue, and ".." denotes the double scalar product.

To find the lowest simple eigenvalue λ_1 and the corresponding eigenelement $\mathbf{u}_1$, we will use the iterative algorithm whose convergence is proved by the following theorem.

Theorem 1. If the sequence of iterations $\{\mathbf{z}^{(k)}\}$ is realized in such a way that

$$\forall \mathbf{v} \in V, \ \langle \mathbf{w}^{(k)}, \mathbf{v} \rangle = (\mathbf{z}^{(k-1)}, \mathbf{v}) \tag{1.2}$$
$$\mathbf{z}^{(k)} = \mathbf{w}^{(k)} / | \mathbf{w}^{(k)} |, \quad k = 1, 2, \ldots \tag{1.3}$$

and $(\mathbf{z}^{(0)}, \mathbf{u}_1) \neq 0$, then

$$\| \mathbf{z}^{(k)} - \mathbf{u}_1 \|_{k \to \infty} \to 0 \tag{1.4}$$

Proof. Since the orthonormalized system of eigenvectors $\{\mathbf{u}_n\}$ is complete /2/, we have

$$\mathbf{w}^{(k)} = \sum_{i=1}^{\infty} a_i^{(k)} \mathbf{u}_i, \quad \mathbf{z}^{(k)} = \sum_{i=1}^{\infty} b_i^{(k)} \mathbf{u}_i \tag{1.5}$$

where $a_i^{(k)}$ and $b_i^{(k)}$ are Fourier coefficients, and the closure equations hold. Noting the orthogonality relations

$$\langle \mathbf{u}_k, \mathbf{u}_l \rangle = \lambda_k \delta_{kl}, \quad (\mathbf{u}_k, \mathbf{u}_l) = \delta_{kl}; \quad k, l = 1, 2, \ldots$$

we see by substituting (1.5) into (1.2) that $a_i^{(k)} \lambda_i = b_i^{(k-1)}, \ i = 1, 2, \ldots$. On the other hand, from

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(1.3) we have

$$b_i^{(k)} = a_i^{(k)} \Big[\sum_{j=1}^{\infty} (a_j^{(k)})^2\Big]^{-1/2}$$

Then,

$$z^{(k)} = f_k \sum_{i=1}^{\infty} \left(\frac{\lambda_1}{\lambda_i}\right)^k a_i^{(0)} u_i, \quad f_k = \left\{\sum_{j=1}^{\infty}\left[\left(\frac{\lambda_1}{\lambda_j}\right)^k a_j^{(0)}\right]^2\right\}^{-1/2}$$

Further, for the norm of the difference we have

$$\| z^{(k)} - u_1 \| = \left[(f_k a_1^{(0)} - 1)^2 \lambda_1 + f_k^2 \sum_{i=2}^{\infty} \left(\frac{\lambda_1}{\lambda_i}\right)^{2k} a_i^{(0)2} \lambda_i\right]^{-1/2}$$

and noting that $f_k |_{k\to\infty} \to 1/a_1^{(0)}$, we finally obtain (1.4). Note that $| w^{(k)} |^{-1}_{k\to\infty} \to \lambda_1$.

To find the subsequent (in magnitude) eigenvalues and the corresponding eigenvectors, we iterate the vector $z^{(0)}$, belonging to the orthogonal complement of the linear envelope of the eigenvectors obtained. Note that, if the spectrum includes multiple eigenvalues, the algorithm remains unchanged, so that it can be used without preliminary information about multiplicity.

2. To find the solution u° of the variational equation

$$\forall v \in V, \quad \langle u, v\rangle = (f, v) \tag{2.1}$$

which appears in the statement of the algorithm, we use the method of geometric imbedding.

We define an open bounded set Ω_0 with boundary Γ_0, such that $\Omega \subset \Omega_0$, $\Gamma_u \subset \Gamma_0$ and there exists on $\Omega_\Delta = \Omega_0 \setminus \Omega$ a solution $v^\circ(w)$ of the boundary-value problem of the theory of elasticity with boundary conditions in the displacements on the part of the boundary coinciding with Γ, and homogeneous conditions in the stresses on the remainder of the boundary

$$\operatorname{div} \sigma(u) = 0, \quad u = w, \quad x \in \Gamma$$

We define the Hilbert space

$$V_0 = \{v \in (H^1(\Omega_0))^n, \ v = 0, \ x \in \Gamma_u\}$$

and introduce the notation

$$\langle u, v\rangle_0 = \int_{\Omega_0} \sigma(u) \cdot\cdot\, e(v)\, d\Omega_0, \quad \langle u, v\rangle_\Delta = \int_{\Omega_\Delta} \sigma(u) \cdot\cdot\, e(v)\, d\Omega_\Delta$$

We consider the boundary-value problem of the theory of elasticity for an inhomogeneous body which occupies the domain Ω_0 and has in Ω the Young's modulus E, and in the domain Ω_Δ the modulus $E_2 = \varepsilon E$, $0 < \varepsilon < 1$, and the constant Poisson's ratio ν in the whole of Ω_0. There exists in space V_0 a unique element $w^\circ \in V_0$ which is the solution of the variational equation

$$\forall v \in V_0, \quad \langle w, v\rangle + \varepsilon \langle w, v\rangle_\Delta = (f, v) \tag{2.2}$$

We define the one-to-one mappings

$$g: V_0 \to V, \quad \forall u \in V_0, \exists v \in V; \quad v = g(u) \Leftrightarrow \forall x \in \Omega, \quad v(x) = u(x) \tag{2.3}$$

$$h: V \to V_0; \quad \forall u \in V, \exists v \in V_0; \quad v = h(u) \Leftrightarrow \forall x \in \Omega\, v(x) = u(x), \quad \forall x \in \Omega_\Delta v = v^\circ(u) \tag{2.4}$$

Note that $\forall u \in V$, $v \in V_0$, $\langle h(u), v\rangle = \langle u, g(v)\rangle$

The fact that the solutions u° and w° of Eqs.(2.1) and (2.2) are close is proved by the following theorem.

Theorem 2. By taking a sufficiently small parameter ε, the solutions of Eqs.(2.1) and (2.2) can be made as close as desired, in the region Ω, so that

$$\forall \delta > 0, \quad \exists \varepsilon > 0, \quad \| u^\circ - g(w^\circ) \| < \delta \tag{2.5}$$

Proof. We put $c = \langle h^\circ, h^\circ\rangle$, where $h^\circ = h(u^\circ)$. Subtracting (2.2) from (2.1) and putting $v = h^\circ - w^\circ$, we obtain

$$\langle h^\circ - w^\circ, h^\circ - w^\circ\rangle = \varepsilon \langle w^\circ, h^\circ\rangle - \varepsilon \langle w^\circ, w^\circ\rangle_\Delta$$

whence $\langle w^\circ, h^\circ\rangle_\Delta \geqslant 0$ On the other hand,

$$\langle h^\circ - w^\circ, h^\circ - w^\circ\rangle = -\varepsilon \langle h^\circ - w^\circ, h^\circ - w^\circ\rangle_\Delta + \varepsilon \langle h^\circ, h^\circ\rangle_\Delta - \varepsilon \langle h^\circ, w^\circ\rangle_\Delta \leqslant \varepsilon c$$

and with $\varepsilon = \delta^2/(2c)$ we obtain (2.5).

We can use the theorem to find the solution of Eq.(2.2) instead of the solution of (2.1); they are close when ε is sufficiently small.

Noting that

$$\forall \mathbf{u}, \mathbf{v} \in V_0, \quad \langle \mathbf{u}, \mathbf{v}\rangle_0 = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle_\Delta$$

we can seek the solution of (2.2) as the expansion

$$\mathbf{w}^\circ = \sum_{n=0}^{\infty} \varkappa^n \mathbf{w}^{(n)}, \quad \varkappa = 1 - \varepsilon \tag{2.6}$$

$$\forall \mathbf{v} \in V_0, \langle w^{(0)}, \mathbf{v}\rangle_0 = (\mathbf{f}, \mathbf{v}), \quad \langle \mathbf{w}^{(n)}, \mathbf{v}\rangle_0 = \langle \mathbf{w}^{(n-1)}, \mathbf{v}\rangle_\Delta$$

We have

$$\langle \mathbf{w}^{(n)}, \mathbf{w}^{(n)}\rangle_0 - \langle \mathbf{w}^{(n-1)}, \mathbf{w}^{(n-1)}\rangle_0 = -\langle \mathbf{w}^{(n)}, \mathbf{w}^{(n)}\rangle - \langle \mathbf{w}^{(n-1)}, \mathbf{w}^{(n-1)}\rangle - \langle \mathbf{w}^{(n)} - \mathbf{w}^{(n-1)}, \mathbf{w}^{(n)} - \mathbf{w}^{(n-1)}\rangle_\Delta \leqslant 0$$

i.e., $\| \mathbf{w}^{(n)} \|_0 < \| \mathbf{w}^{(n-1)} \|_0$, and series (2.6) is convergent in the norm $\| \mathbf{u} \|_0 = \langle \mathbf{u}, \mathbf{u} \rangle_0^{1/2}$ при $| \varkappa | < 1$.

To sum up, our approach reduces the problem of finding the natural frequencies and shapes of the oscillations of a body of complex geometrical shape to an iterative sequence of static problems for a body of simpler shape in which the original body is embedded. Here, we took as the homogeneous elastic canonical domains solids of revolution, and the solutions in the canonical domains were obtained semi-analytically by the finite-element method /3/.

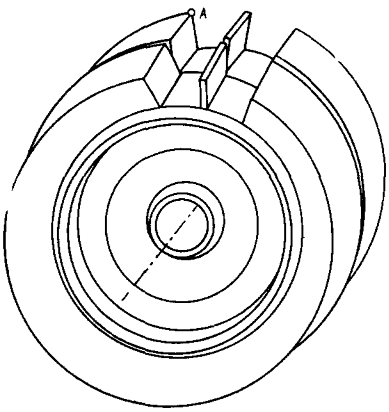

Fig.1

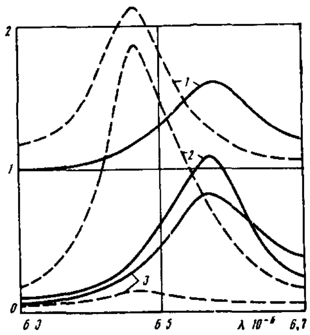

Fig.2

3. The numerical computations were made for a body in the form of a disc of variable thickness with blades mounted along its periphery. The 12 lowest natural frequencies and the corresponding spatial shapes of the oscillations were found. The disc geometry and the shape of the oscillations corresponding to the lowest natural frequency are shown in Fig.1 (for clarity, we show 1 of the 23 blades).

The natural shapes of the oscillations of an elastic body were used as basis functions for studying the forced steady-state oscillations of a visco-elastic body of complex geometry. The study was made in the context of linear Boltzmann-Volterra hereditary theory. The relaxation kernel was the Rzhanitsyn three-parameter kernel /4/.

The broken curves of Fig.2 are the amplitude-frequency responses (AFR) of the r-, z- and φ-components of the displacements of point A of the body, see Fig.1. Curves *1-3* refer to the values $u_r \times 10^3$, $u_z \times 10^2$, $u_\varphi \times 10^3$. The continuous curves are the AFR's of the same point A when one adjacent blade is missing. The blade mass is 0.91% of the total disc mass. It can be seen that it takes only a slight change in the construction to radically change the AFR shape.

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